Tutorial 1

Resilient Communication Systems
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Released: October 13th, 2025.

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RESILIENT COMMUNICATION SYSTEMS

Preliminaries in Linear Algebra (I)

Problem 1

Consider the following two square matrices assuming $x \in \mathbb{R}^+$:

$$\mathbf{A} = \begin{bmatrix} x & -x \\ -x & x \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} -x & 2x \\ 2x & x \end{bmatrix}$$

a) Find the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$, respectively, as a function of x.

b) Derive the determinant and trace of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{AB}$, and $\mathbf{A}+\mathbf{B}$. What is the relationship between the eigenvalues and the determinant and the trace? Show that $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$ and trace$(\mathbf{A}+\mathbf{B})$ =trace$(\mathbf{A})$+trace$(\mathbf{B})$.

c) Prove that trace$(\mathbf{AB})$ = trace$(\mathbf{BA})$ holds for all square matrices.

d) Which of the following statements hold for $\mathbf{C} = \mathbf{A}$ and $\mathbf{C} = \mathbf{B}$, respectively (determine the value of x for the statement to hold true)?

   1) $\mathbf{C}$ is positive (semi-)definite.

   2) $\mathbf{C}$ is Hermitian.

   3) $\mathbf{C}$ has full rank.

   4) $\mathbf{C}^{-1}$ exists.

   5) $\mathbf{C}$ is unitary.

e) Show that $|\mathbf{I}+\mathbf{AB}| = |\mathbf{I}+\mathbf{BA}|$.

f) Assume that $\tilde{\mathbf{B}} = n\mathbf{B}$ where $n \in \mathbb{R}^+$. For which values of n and x, $|\tilde{\mathbf{B}}^{-1}| = 1/|\mathbf{B}|$?

g) Compute $\text{rank}(\mathbf{A})$, $\text{rank}(\mathbf{B})$, and $\text{rank}(\mathbf{AB})$. Verify that rank$(\mathbf{AB}) \leq$ min$\{\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})\}$ hold.

h) List the axioms of a function $||\cdot|| : \mathbb{C}^{M\times N} \to \mathbb{R}$ must satisfy, such that it is a matrix norm on $\mathbb{C}^{M\times N}$.

Problem 2

Consider the 2×3 matrices:

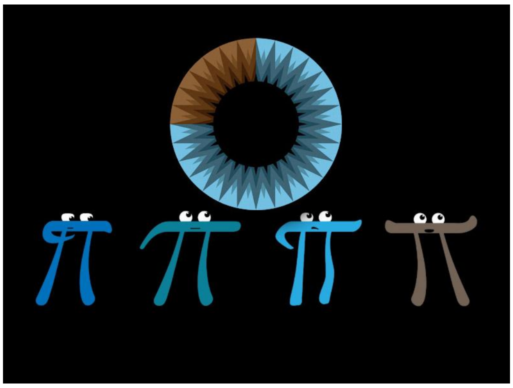
$$\mathbf{A} = \begin{bmatrix} 1 & 0.5 & -0.5 \\ -1 & 1 & -1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} -1 & 1 & -1 \\ -0.5 & 0.5 & 1 \end{bmatrix}$$

a) Specify the rank of $\mathbf{A}$ and $\mathbf{B}$.

b) Compute the singular value decomposition of $\mathbf{A}$ such that $\mathbf{A} = \mathbf{U\Sigma V}$ with unitary matrices $\mathbf{U}$ and $\mathbf{V}$ and the diagonal matrix of singular values $\mathbf{\Sigma}$.

c) Find the QL decomposition of the matrix $\mathbf{B}$ ($\mathbf{B} = \mathbf{QL}$ with $\mathbf{Q}$ being unitary and $\mathbf{L}$ being lower-triangular).

If you would like to gain a deeper, visual understanding of key linear algebra concepts like determinants, eigenvalues, and eigenvectors, I highly recommend checking out the **3Blue1Brown** YouTube channel. Their **Essence of Linear Algebra** series offers beautifully illustrated, intuitive explanations of these topics, making them much easier to grasp. You can start with the following videos:
The determinant
Eigenvectors and eigenvalues



3Blue1Brown YouTube channel

Carl Friedrich Gauss (1777–1855): Key Contributor to Linear Algebra

Carl Friedrich Gauss, often called the "Prince of Mathematicians," made significant contributions to many fields of mathematics, including linear algebra, which is foundational to modern-day communications, including MIMO systems.

Gauss's work on matrices and quadratic forms laid the groundwork for many concepts in linear algebra, including eigenvalues and eigenvectors. In particular, his study of quadratic forms (expressions involving sums of squares of variables) contributed to what would later be formalized as the eigenvalue problem. Quadratic forms are central to understanding how matrices act on vectors, which leads directly to the idea of eigenvalues and eigenvectors.

Johann Carl Friedrich Gauss
Born: 30 April 1777
Died: 23 February 1855 (aged 77)

While Gauss did not explicitly develop the theory of eigenvalues, his work in matrix theory, particularly the diagonalization of symmetric matrices, is directly related to the modern concept of finding eigenvalues. Symmetric matrices, when diagonalized, reveal their eigenvalues along the diagonal, a technique commonly used in physics and engineering.

In summary, while Gauss didn't directly formalize the theory of eigenvalues, his pioneering work on matrices, quadratic forms, and diagonalization was instrumental in shaping the early development of eigenvalue theory.